1 Stone Ecliptic

You are given the map of Calakmul, a Maya archaeological site in the Mexican state of Campeche. On the main square (Gran Plaza) there are several buildings; in particular, building VI (Estructura VI) and building IV (Estructura IV). The latter is the complex of three temple towers.

It is known that if one looks from building VI to building IV, on the days of equinoxes and solstices, the rising Sun touches the tops of the respective towers (marked). The latitude of Calakmul is 18° North.

Estimate the height of the towers relative to the observation point. Consider the Sun as a point source.

Please find calakmul.jpg to zoom in.

Solution:

The middle tower corresponds to the position of the Sun on the equinoxes. However, it can be seen that it is not located exactly east of the observation point, but approximately 14° south. This is because the Sun is observed not at the moment of crossing the horizon, but at a certain height corresponding to the height of the tower.

Let us find the height of the Sun at the moment of "rise" above the tower. From the parallactic triangle (taking into account $\delta = 0°$):

$$\sin\delta = \sin\varphi\sin h - \cos\varphi\cos h\cos A,$$

$$\tan h = \frac{\cos A}{\tan\varphi} \quad \Rightarrow \quad h = 36.5°.$$

The distance from the observation point to the tower is $L = 80$ m. The height of the tower

$$H = L\tan h = 80\ \text{m}\cdot\tan 36.5° = 59\ \text{m}.$$

The difference in azimuths between the middle and side towers is approximately $23° - 24°$, which corresponds to the difference in the azimuths of sunrise at the equinoxes and solstices at the latitude of Calakmul. We can consider the towers to be the same height.

Note. "Flat" approximation gives

$$h = 14°\cdot\tan(90° - \varphi) = 43°,$$
$$H = L\tan h = 75\ \text{m}.$$

The answer seems to be somewhat exaggerated: the height of building I, the main building in Calakmul, is estimated at 55 m. However, the calculation results are quite sensitive to the determination of the observation point and azimuth.

Marking Scheme:

- Distance L between buildings — **2 pt.**
- Azimuth ΔA of the tower — **4 pt.**
- Altitude h of the Sun — **8 pt.**
  In particular, a flat approximation — 4 pt.
- Height h of the tower — **2 pt.**
- Heights of other towers (explanation or calculation) — **4 pt.**

2 RR Lyrae

In the field of the globular star cluster M4, observed by the GAIA space telescope, a number of RR Lyrae variables were found. The equatorial coordinates of the center of M4 are $\alpha_0 = 16^{\mathrm{h}}\,23^{\mathrm{m}}\,35.22^{\mathrm{s}}$, $\delta_0 = -26^\circ\,31'\,32.7''$. You are given data on the RR Lyrae variables:

- the equatorial coordinates (α, δ) of objects,
- the parallax ϖ in milliarcseconds,
- the components $(\mu_\alpha \cos\delta, \mu_\delta)$ of their proper motion,
- the period of pulsation,
- the average magnitudes in G, G_{BP} and G_{RP} bands,
- the average radial velocity $\langle V_r \rangle$,
- the amplitude X_G of pulsation in G band and its error ΔX_G,
- the absorption A_G in G band.

a) Determine which stars do not belong to the cluster.
b) Find the period-luminosity relation for RR Lyrae variables of the M4 cluster.
c) Estimate the distance to the cluster.
d) Estimate the mass of the cluster.

Source ID Name	α, °	δ, °	ϖ mas	$\mu_\alpha \cos\delta$ mas/yr	μ_δ mas/yr	P days	$\langle G \rangle$ mag	$\langle G_{BP} \rangle$ mag	$\langle G_{RP} \rangle$ mag	$\langle V_r \rangle$ km/s	X_G mag	ΔX_G mag	A_G mag
60...496	245.7225	−27.4102	0.303	−6.089	−2.476	0.767	14.27	14.81	13.55		0.57	0.02	1.21
60...168	245.8310	−26.6659	0.570	−12.015	−19.382	0.507	13.00	13.45	12.43	72.38	0.89	0.07	0.84
...	...	...	...	...	...	...	...	...	...	...	...	...	...

The full data table can be found in M4_0.csv file.

Solution:

a) Let us calculate mean values and standard deviations of the kinematic parameters:

Parameter	Mean	St. dev.
ϖ, mas	0.542	0.116
$\mu_\alpha \cos\delta$, mas/yr	−11.155	4.596
μ_δ, mas/yr	−17.715	4.000
$\langle V_r \rangle$, km/s	69.35	8.62

We should discard the stars most deviating from the mean by 3 or all 4 parameters:

ID	$\Delta\varpi/\sigma_\varpi$	$\Delta(\mu_\alpha \cos\delta)/\sigma_{\mu_\alpha}$	$\Delta\mu_\delta/\sigma_{\mu_\delta}$	$\Delta V_r/\sigma_{V_r}$
6045280135437770496	2.06	1.10	3.81	—
6045651495491644928	2.20	4.32	0.45	0.90
6045438018442520576	3.57	1.03	2.43	3.24

Then we calculate absolute magnitudes taking extinction in the G band into account:

$$M_G = m_G + 5 + 5\log\varpi - A_G.$$

Now we can continue the selection process. Since the stars in a globular cluster have approximately the same age, the period of pulsation and the absolute magnitude should also be quite similar:

ID	$\Delta\varpi/\sigma_\varpi$	$\Delta(\mu_\alpha\cos\delta)/\sigma_{\mu_\alpha}$	$\Delta\mu_\delta/\sigma_{\mu_\delta}$	$\Delta V_r/\sigma_{V_r}$	$\Delta M_G/\sigma_{M_G}$	$\Delta P/\sigma_P$
6045502305516138368	1.83	0.50	0.83	1.32	2.59	1.09
6045488767778755456	2.48	0.46	1.51	0.19	0.42	0.40
6045466055990618496	0.84	3.28	0.92	-	1.46	3.53

After the second round of selection, there is no parameter with a deviation greater than 3σ.

b) The parameters of the period-luminosity relation $\lg P - M_G$ are obtained using least squares:

$$M_G = -1.657\lg P\,[\mathrm{d}] + 0.289.$$

c) The distance to the cluster is estimated from the mean parallax:

$$R = \frac{1\ \mathrm{pc}}{\varpi\,['']} = \frac{1\ \mathrm{kpc}\cdot\mathrm{mas}}{0.561\ \mathrm{mas}} = 1.78\ \mathrm{kpc} \approx 1.8\ \mathrm{kpc}.$$

d) To estimate the mass of a cluster, it is necessary to determine the velocities of the stars relative to the center of the cluster. As a first approximation,

$$\vec{v}_i = \vec{V}_i - \vec{V}_{\mathrm{GC}},$$

$$\text{where}\quad \vec{V}_i \simeq V_{r,i}\cdot\vec{e}_r + \mu_{\alpha,i}\cos\delta_i\cdot R\cdot\vec{e}_\alpha + \mu_{\delta,i}\cdot R\cdot\vec{e}_\delta,$$

$$\vec{V}_{\mathrm{GC}} \simeq \left\langle\vec{V}_i\right\rangle.$$

According to the virial theorem,

$$\left\langle v_i^2\right\rangle \sim \frac{GM_{M4}}{r_{\mathrm{GC}}}.$$

The radius of the cluster may be estimated as

$$r_{\mathrm{GC}} \simeq R\cdot\max_i\left\{\sqrt{(\alpha_i-\alpha_0)^2\cos\delta_0 + (\delta_i-\delta_0)^2}\right\}.$$

Finally, the mass of the cluster

$$M_{M4} \sim \frac{\langle v_i^2\rangle\cdot r_{\mathrm{GC}}}{G} \sim \frac{(2.4\ \mathrm{km/s})^2\times 90\ \mathrm{pc}}{G} \sim 10^5 M_\odot.$$

Marking Scheme:

- Star selection — **6 pt:**
  - using 4 parameters with 2 iterations — full points.
  - only parallax is used — **2 pt.**
  - 2 iteration with parallax or parallax and some parameters — **4 pt.**
- Coefficient — **6 pt.**
  In particular, correct coefficient with incorrect sampling — 3 pt.
- Distance to the cluster — **2 pt.**
- Mass of the cluster (any appropriate method) — **6 pt.**
  In particular,
  - Speed of stars relative to the center of the cluster is ignored — 3 pt.
  - Inadequate final answer — $\leqslant 1$ pt.

3 Statistical Parallax

You are given data on some stars of an open cluster associated with a stellar stream:

- the equatorial coordinates (α, δ) of objects,
- the components $(\mu_\alpha \cos\delta, \mu_\delta)$ of their proper motion,
- the radial velocity V_r.

Consider this cluster to consist only of the specified stars.

a) Estimate the coordinates of the apex/radiant for the cluster objects.
b) Evaluate the parallaxes of individual objects, plot the distribution of parallaxes.
c) Determine the statistical parallax of the cluster.

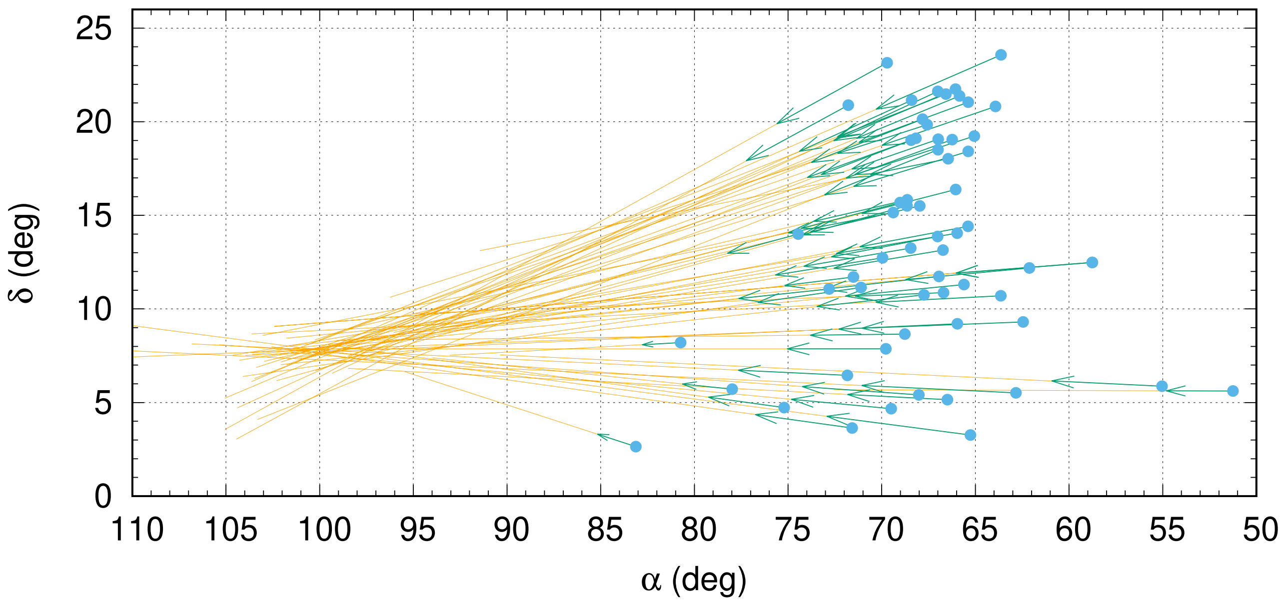
Name	α	δ	$\mu_\alpha \cos\delta$, mas/yr	μ_δ, mas/yr	V_r, km/s
HD 27990	$4^h25^m48^s$	$+18°01'02''$	75	-14	40.5
HD 27835	$4^h24^m13^s$	$+16°22'44''$	90	-22	39.5
...	...	...	...	...	...

The full data table can be found in statistical_parallax.csv file.

Solution:

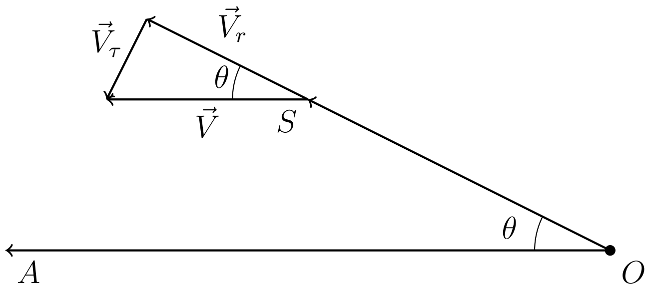
a) To determine the apex we should plot the vectors of the proper motion of all the objects. For simplicity, we plot the diagram on (α, δ) plane not taking into account the sphericity of this part of the celestial sphere, so that $\mu = \sqrt{\mu_\delta^2 + \mu_\alpha^2}$.

The picture below shows the distribution of proper motions of the stars. The scale of the green vectors indicating the proper motions is arbitrary.



To determine the apex we lenghten the proper motion vectors and determine the coordinates of their intersection. High proper motion and the extent of the cluster show that it is quite close to us. We can approximately locate the apex at a point with coordinates ($\alpha = 101°,\ \delta = 8°$).

b) Assuming that all stars move at the same velocities, we are able to determine the parallax ϖ of each star. Let θ be the angle between the direction to the star and to the apex A of the cluster.



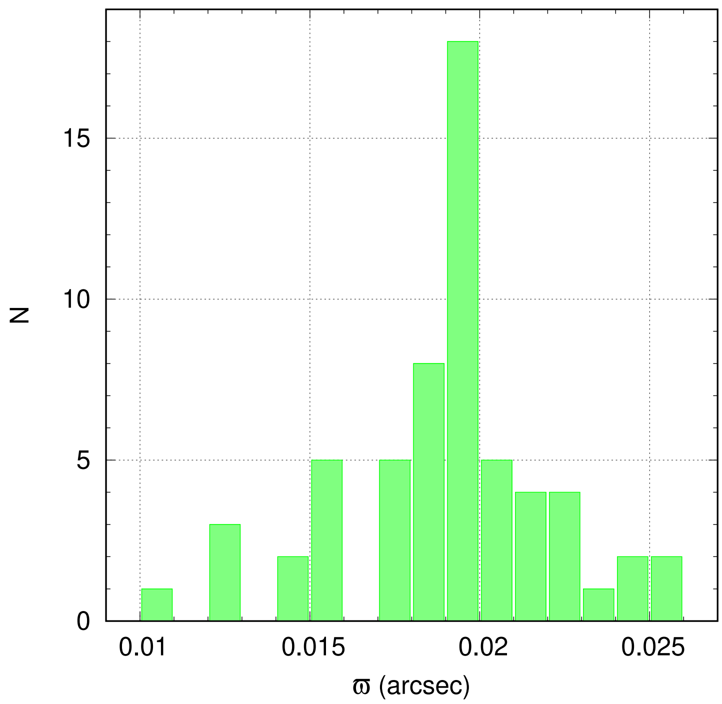
For the i-th star

$$V_{\tau,i} = 4.74\mu_i \cdot \frac{1}{\varpi_i},$$

where $\mu_i = \sqrt{\mu_{\delta,i}^2 + (\mu_{\alpha,i}\cos\delta_i)^2}$. Therefore, the parallax ϖ_i can be estimated by the formula

$$\varpi_i = \frac{4.74\mu_i}{V_{\tau,i}} = \frac{4.74\mu_i}{V_{r,i}\tan\theta_i}\,.$$

The distribution of parallaxes is shown on the histogram below:



c) The distribution of parallaxes is quite symmetric, so we may propose the mean parallax to be the estimation of the cluster parallax:

$$\langle \varpi \rangle = \frac{1}{N} \sum_{i=1}^{N} \varpi_i = 0.019''.$$

This value corresponds to a distance $\approx 52\,\mathrm{pc}$.

The table provided data for the sample of the stars of the Hyades cluster and stream. The current parallax estimate for this cluster is about $21''$. Thus, our results are rather realistic.

Marking Scheme:

- Statistical approach: drawing of the cluster, lines of proper motion, looking for their intersection or least squares, analysis, etc. — **5 pt.**
- Coordinates of the apex — **1 pt** + **1 pt.**
- Formula for the distance or parallax within the statistical parallax method — **5 pt.**
- Table or column of individual parallaxes — **3 pt.**
- Plot (histogram) of the parallax distribution with readable binning — **3 pt.**
- Parallax of the cluster — **2 pt.**

4 Nearby Cluster

The data describe one of the nearby galaxy clusters (Eftekhari et al., 2022):

- (α, δ) are the equatorial coordinates of the galaxy,
- R_e is the effective radius of the galaxy,
- $M_{r,e}$ is the absolute magnitude in r band inside the effective radius,
- $\mu_{r,e}$ is the average surface brightness in r band inside the effective radius measured in magnitudes per square arcsecond,
- q is the axis ratio of the projection of galaxy on the plane of the sky.

Determine the following parameters of the galaxy cluster:

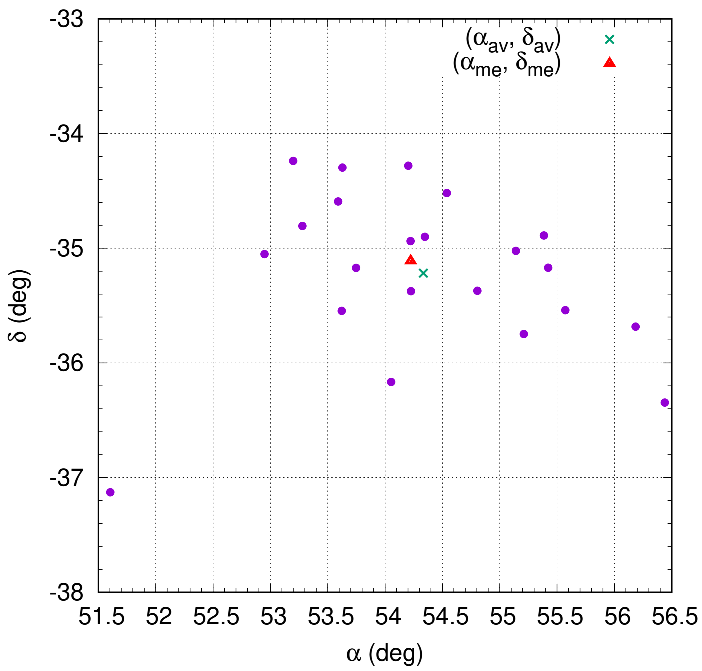
a) the coordinates of the center of the cluster,
b) the constellation where the center of the cluster is located,
c) the distance to the cluster,
d) the linear dimensions of the cluster in the plane of the sky in α and δ directions.

ID	α, °	δ, °	R_e, ″	$M_{r,e}$	$\mu_{r,e}$	q
22_D244	51.6043	−37.1278	8.5	−15.32	21.60	0.64
16_D417	52.9485	−35.0514	19.8	−16.14	22.79	0.76
...	...	...	...	...	...	...

The full data table can be found in `nearby_cluster.csv` file.

Solution:

a) To determine the center of the cluster, we will plot the distribution of galaxies at first:



The graph shows that the galaxy with the lowest right ascension is located at a considerable distance from other objects, so its coordinates will significantly shift the estimation of the coordinates of the cluster center by calculating the arithmetic mean

$$\alpha_{\text{av}} = 54.33^\circ, \quad \delta_{\text{av}} = -35.22^\circ.$$

It would be more correct to calculate the median values of the coordinates, rather than the mean values. After having the arrays ordered, we calculate the half-sum of the central elements (the number of galaxies is even):

$$\alpha_{\text{me}} = 54.22^\circ, \quad \delta_{\text{me}} = -35.11^\circ.$$

b) The coordinates of the center belong to Fornax. The table provided data on the galaxies of the Fornax Cluster, the second closest galaxy cluster.

c) As surface brightness, effective radius and axis ratio are given, we can determine the apparent magnitude of each galaxy. The area of elliptic projection of the galaxy on the plane of the sky is $\pi R_e^2 q$. The apparent magnitude of the galaxy inside the effective radius is

$$m_{r,e} = \mu_{r,e} - 2.5 \lg(\pi R_e^2 q).$$

Next, let us obtain the distance to the galaxies from the apparent and absolute magnitudes. The cluster is far from the galactic plane, so we will ignore the interstellar extinction:

$$m_{r,e} = M_{r,e} - 5 + 5 \lg r[\text{pc}],$$
$$r\,[\text{pc}] = 10^{0.2(m_{r,e} - M_{r,e} + 5)}.$$

As an estimate of the distance to the cluster, we take the average value: $r = 20.0$ Mpc.

Note. The current estimate of the distance to the Fornax Cluster is 19.7 Mpc.

d) Since the galaxy with the lowest right ascension still belongs to the cluster, it should be taken into account when estimating the size of the cluster. We should note that the angular dimension of the cluster in α direction also depends on the declination of the cluster:

$$d_\alpha = (\alpha_{\max} - \alpha_{\min}) \cos \delta_{\text{me}} = (56.4931^\circ - 51.6043^\circ) \cdot \cos(-35.11^\circ) = 4.0^\circ.$$

The angular dimension in δ direction is

$$d_\delta = \delta_{\max} - \delta_{\min} = -34.2387^\circ + 37.1278^\circ = 2.9^\circ.$$

The linear dimensions are

$$D_\alpha = r \cdot 2 \tan \frac{d_\alpha}{2} = 1.7 \text{ Mpc},$$
$$D_\delta = r \cdot 2 \tan \frac{d_\delta}{2} = 1.2 \text{ Mpc}.$$

Marking Scheme:

- Coordinates of the center of the cluster — **5 pt:**
  - only the answer is given — 1 pt;
  - OR arithmetic mean of the maximum and minimum values of the coordinates, the outlier is NOT mentioned — 2 pt;
  - OR arithmetic mean of the maximum and minimum values of the coordinates, the outlier IS mentioned — 3 pt,
  - OR arithmetic mean of the coordinates of the whole sample — 4 pt,
  - OR median value of the coordinates of the whole sample — full points.

  Penalties:

  - Excessive accuracy of the answer — −2 pt.
  - Computational error — −2 pt.
- Constellation — **3 pt:**
  - Fornax — full points,
  - OR Eridanus — 2 pt.
- Distance estimation — **7 pt:**
  - 20.0 ± 0.1 Mpc — full points,
  - OR 20 ± 5 Mpc — 5 pt.

  Penalties:

  - Distance is estimated based on the two outermost galaxies — −2 pt.
  - Axis ratio is neglected — −2 pt.
  - Excessive accuracy of the answer — −2 pt.
- Linear dimensions of the cluster.
  - Angular dimensions estimation — **2 pt.**
  - Declination is taken into account — **1 pt.**
  - Linear dimensions — **2 pt.**

  Penalties:

  - Excessive accuracy of the answer — −2 pt.